ELSEVIER

Linear Algebra and its Applications 338 (2001) 1–17

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

Algebraic aspects of the discrete KP hierarchy ☆

R. Felipe [a,b], F. Ongay [b,*]

[a] ICIMAF, Havana, Cuba

[b] CIMAT, s/n Mineral de Valenciana, Apdo. Postal 402, Guanajuato, GTO, Mexico

Received 6 February 2001; accepted 7 April 2001

Submitted by R.A. Brualdi

Abstract

We discuss some algebraic properties of the so-called discrete KP hierarchy, an integrable system defined on a space of infinite matrices. We give an algebraic proof of the complete integrability of the hierarchy, which we achieve by means of a factorization result for infinite matrices, that extends a result of M. Adler and P. Van Moerbeke [Commun. Math. Phys. 203 (1999) 185; 207 (1999) 589] for the case of (semi-infinite) moment matrices, and that we call a *Borel decomposition*. © 2001 Elsevier Science Inc. All rights reserved.

AMS classification: 15A30; 37K10

Keywords: Discrete KP hierarchy; Borel decomposition; Infinite dimensional integrable systems

1. Introduction

Discrete analogs of the KP hierarchy were recently introduced and studied by Adler and Van Moerbeke [2,3]. Here, in the standard Lax operator the role of ∂_x is taken by the so-called shift matrix Λ, and the coefficients are taken to be infinite diagonal matrices, and in these works, Adler and Van Moerbeke obtained many interesting results, for instance in relation with the representation theory of the τ-functions of the classical KP hierarchy.

Nevertheless, the explicit use made in those works of some quite specific analytic techniques—such as the theory of positive measures and orthogonal polynomials

☆ Partially supported by CONACYT, Mexico, project 28-492E, and Mexico–Cuba 2000.

* Corresponding author.

E-mail addresses: raulf@cimat.mx (R. Felipe), ongay@cimat.mx (F. Ongay).

PII: S 0 0 2 4 - 3 7 9 5 (0 1) 0 0 3 6 5 - 2

associated to the measure and some moment matrix—does not appear to be the best way to develop a general study of the *space of solutions* of these hierarchies.

On the other hand, the algebraic approach due to Mulase [5] for the study of the KP equations—which we follow here—seems to us ideal for this task, and as is well known, the key point in Mulase's theory is a factorization theorem, akin to the celebrated Birkhoff decomposition of loop groups. Thus, in this paper we state and prove some results on a type of factorization of matrices, which by analogy to the result of Adler and Van Moerbeke for moment matrices (loc. cit.), we call a *Borel decompostion* (some authors call this type of decomposition a *Gauss decomposition*); we then show that Mulase's results remain valid almost verbatim in this context. In particular, this approach allows us to consider on an almost equal footing the cases of *semi-infinite* and *bi-infinite* matrices, the discussion of the latter case being, to the best of our knowledge, new to the literature on matrix integrable systems. (We should add that we are aware of a recent paper by Adler [1], where a factorization of the semi-infinite moment matrix is discussed, but unfortunately have not had access to it.)

Furthermore, we believe that this approach is the right one to relate the discrete KP hierarchy to harmonic maps, as is done for instance in [4] for the finite dimensional case, and we hope to do this in a forthcoming paper.

A brief description of the paper is as follows: In Section 2, we discuss some of the algebraic properties of the space of infinite dimensional matrices, and prove the factorization results. In Section 3, we consider the discrete KP hierarchy, and study its integrability in the sense of Frobenius.

2. The structure of the space of infinite matrices

Let $\mathscr{D}_{-\infty}^{\infty}$ and $\mathscr{D}^{\infty}$ denote the sets of all bi-infinite and semi-infinite matrices, that is elements of $\mathbb{R}^{\mathbb{Z}\times\mathbb{Z}}$ and $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$, respectively. For the remainder of this paper, we will use the common notation $\mathscr{D}$ for both, $\mathscr{D}_{-\infty}^{\infty}$ and $\mathscr{D}^{\infty}$, whenever the arguments apply to both of these spaces, pointing out the specific instances, where one has to distinguish between the two cases.

By Λ we denote the *shift matrix*, having 1's in the first upper diagonal and 0's elsewhere, and consider its transpose Λ^{T}, and let I be the identity matrix in $\mathscr{D}$. Now if A, B are any two matrices in $\mathscr{D}$ having only one non-vanishing diagonal, it is clear that the usual formula for the product of matrices makes sense for this kind of matrices, and so AB is well defined. Therefore, we can consider the iterates of the shift matrix and its transpose, Λ^n and Λ^{T^n}, and observe that, for a diagonal matrix A, the product $A\Lambda^n$ (respectively, $A\Lambda^{\mathrm{T}^n}$) is a matrix having 0's everywhere, except possibly in the nth upper (respectively, lower) diagonal, where it has the same entries as A (which explains the name "shift matrix"). Therefore, we see that $\mathscr{D}$ can be identified with the set of formal bi-infinite series

$$A = A_0 + \sum_1^\infty A_n \Lambda^n + \sum_1^\infty B_n \Lambda^{\mathrm{T}n}, \tag{1}$$

and in fact that $\mathscr{D}$ has the structure of a left-module over the ring of diagonal matrices, "freely" generated by the powers of Λ and Λ^{T}. Furthermore, from expression (1) we have an obvious direct sum decomposition of $\mathscr{D}$ given by

$$\mathscr{D} = \mathscr{D}_- \oplus \mathscr{D}_0 \oplus \mathscr{D}_+, \tag{2}$$

where $\mathscr{D}_0$ are the (principal) diagonal matrices, and $\mathscr{D}_+$ (respectively, $\mathscr{D}_-$) the strictly upper (respectively, lower) triangular matrices.

Remark. We will sometimes refer to the elements of $\mathscr{D}$ as *operators*, since they obviously can act on infinite row vectors, although we will not actually use this property in this work.

An important difference between $\mathscr{D}_{-\infty}^{\infty}$ and $\mathscr{D}^{\infty}$ is that for the former $\Lambda^{\mathrm{T}} = \Lambda^{-1}$, while for the latter Λ^{T} is only a left inverse; that is $\Lambda^{\mathrm{T}}\Lambda = I$, but $\Lambda\Lambda^{\mathrm{T}}$ is the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \\ \vdots & \vdots & & \ddots \end{pmatrix}.$$

On the other hand, we have the following easily verified commutation properties $A\Lambda = \Lambda A^{(1)}$, where $A^{(1)}$ is given as follows.

For $\mathscr{D}_{-\infty}^{\infty}$, if

$$A = \begin{pmatrix} \ddots & \vdots & & & \\ \cdots & a_{-1} & 0 & 0 & \cdots \\ \cdots & 0 & a_0 & 0 & \cdots \\ \cdots & 0 & 0 & a_1 & \cdots \\ & \vdots & & & \ddots \end{pmatrix},$$

then

$$A^{(1)} = \begin{pmatrix} \ddots & \vdots & & & \\ \cdots & a_{-2} & 0 & 0 & \cdots \\ \cdots & 0 & a_{-1} & 0 & \cdots \\ \cdots & 0 & 0 & a_0 & \cdots \\ & \vdots & & & \ddots \end{pmatrix};$$

whereas for $\mathscr{D}^{\infty}$, if

$$A=\begin{pmatrix} a_1 & 0 & 0 & \cdots \\ 0 & a_2 & 0 & \cdots \\ 0 & 0 & a_3 & \cdots \\ \vdots & & & \ddots \end{pmatrix}, \quad \text{then} \quad A^{(1)}=\begin{pmatrix} * & 0 & 0 & \cdots \\ 0 & a_1 & 0 & \cdots \\ 0 & 0 & a_2 & \cdots \\ \vdots & & & \ddots \end{pmatrix},$$

where $*$ here means that we can put an arbitrary number. Similarly, we have $A\Lambda^{\mathrm{T}} = \Lambda^{\mathrm{T}}A^{(-1)}$; and more generally, to any diagonal matrix A we can define *associated* matrices $A^{(n)}$, for $n \in \mathbb{Z}$, where $A^{(0)} = A$, and for $n > 0$, $A\Lambda^n = \Lambda^n A^{(n)}$, $A\Lambda^{\mathrm{T}n} = \Lambda^{\mathrm{T}}A^{(-n)}$.

The product of two infinite matrices is certainly a much more delicate thing to define, because in general it will involve divergent series. Several options have been considered in the literature, such as $\mathscr{G}l_\infty$ (infinite matrices with "finite support"), and $\hat{\mathscr{G}}l_\infty$ (infinite band matrices), but none of these is general enough for our purposes, because we want to consider some matrices having an infinite number of (possibly) non-zero diagonals.

On the other hand, it is certainly possible to multiply such matrices: In fact, it is easy to see that the submodules $\mathscr{D}_0 \oplus \mathscr{D}_-$ and $\mathscr{D}_0 \oplus \mathscr{D}_+$ are closed under multiplication, because the products involved here are all finite. Moreover, we have the following easy technical lemma.

Lemma 1. *The inverse of an operator $M \in \mathscr{D}_0 \oplus \mathscr{D}_-$ (respectively, $\mathscr{D}_0 \oplus \mathscr{D}_+$), if it exists, is also an operator in $\mathscr{D}_0 \oplus \mathscr{D}_-$ (respectively, $\mathscr{D}_0 \oplus \mathscr{D}_+$).*

Furthermore, M is invertible if and only if M_0 is invertible, and then $M^{-1} = M_0^{-1} + \hat{M}$, with $\hat{M}$ strictly triangular.

Proof. The proof is in fact constructive: if one assumes this form for the inverse, then one can write a system of algebraic equations that can be solved recursively, to explicitly construct an inverse; then use the uniqueness of inverses. Finally, the second assertion is included in these computations. □

Therefore, the matrices in $\mathscr{D}_0 \oplus \mathscr{D}_-$ (respectively, $\mathscr{D}_0 \oplus \mathscr{D}_+$) satisfying the condition that their principal diagonal M_0 is invertible, can be regarded as infinite dimensional Lie groups. It is also clear that their Lie algebras can be taken to be the corresponding full submodules, $\mathscr{D}_0 \oplus \mathscr{D}_-$ and $\mathscr{D}_0 \oplus \mathscr{D}_+$.

Still more generally, if P and M are matrices such that their projections on $\mathscr{D}_+$ (respectively, $\mathscr{D}_-$) are finite band matrices, then the product is well defined and of the same type. Finally, we can single out another sub-class of $\mathscr{D}^\infty$, for which the product also exists; namely, products of the form $(M_- + M_0 + M_+)(P_- + P_0 + P_+)$ are here well defined, as long as M_+ and P_- are finite band matrices.

The problem, of course, arises when we try to multiply an arbitrary matrix $M_- \in \mathscr{D}_-$ with a matrix $M_+ \in \mathscr{D}_+$ or vice versa. Nevertheless, for us it is essential to consider such products because, in fact, we shall concentrate our attention on a special sub-class of $\mathscr{D}$, of matrices satisfying the condition

$$M = H + H_- H_0^{-1} H_+ \quad \text{for some invertible } H \in \mathscr{D}.$$

Let us denote this set $\mathscr{M}$, and remark first of all that the key property of these matrices—as an easy computation shows—is that every such M admits a decomposition of the form

$$M = (I + H_- H_0^{-1})(H_0 + H_+),$$

which we call a *Borel decomposition* or *factorization* of M.

Remark. Observe that we have a (highly non-linear) operator of "Borel symmetrization", sending a matrix H to the Borel decomposable matrix $P = H + H_- H_0^{-1} H_+$. The problem of deciding what are the "right type of matrices" can be restated as the problem of characterizing its domain and range.

Actually, to obtain meaningful results about integrable systems, we will still need to impose several additional conditions on the matrices, and these will be described later on; but let us next describe some cases where the Borel decomposition is valid, starting with the following basic result.

Proposition 1. *The Borel decomposition is unique.*

Proof. Assume that $M = (I + H_- H_0^{-1})(H_0 + H_+) = (I + K_- K_0^{-1})(K_0 + K_+)$. Then it follows that

$$(I + K_- K_0^{-1})^{-1}(I + H_- H_0^{-1}) = (K_0 + K_+)(H_0 + H_+)^{-1}.$$

Since, by Lemma 1, the LHS is lower triangular and the RHS is upper triangular, both sides are diagonal; but the diagonal in the LHS is I. Thus the RHS gives $H_0 = K_0$, $H_+ = K_+$, and therefore $H = K$. $\square$

Now, for semi-infinite matrices we have a good description of the matrices admitting a Borel decomposition.

Proposition 2. *Let $P \in \mathscr{D}^\infty$; let P_i denote the principal $i \times i$ submatrix of P, i.e., the upper left corner $i \times i$ submatrix, and $|P_i|$ its determinant. Then P admits a Borel decomposition if and only if $|P_i| \neq 0$ for all $i \geqslant 0$. Furthermore, H_0 then has diagonal elements $h_{ii} = |P_i|/|P_{i-1}|$ (by convention $|P_{-1}| = 1$).*

Proof. Indeed, if a 2×2-matrix

$$P = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$$

admits the decompositon $P = H + H_- H_0^{-1} H_+$, then by a straightforward calculation one shows that

$$H_- = \begin{pmatrix} 0 & 0 \\ a_{10} & 0 \end{pmatrix}, \quad H_+ = \begin{pmatrix} 0 & a_{01} \\ 0 & 0 \end{pmatrix}$$

and

$$H_0 = \begin{pmatrix} a_{00} & 0 \\ 0 & \det P/a_{00} \end{pmatrix} = \begin{pmatrix} |P_0| & 0 \\ 0 & |P_1|/|P_0| \end{pmatrix}$$

as desired. We now can use induction to solve the question for finite matrices. The key point of the proof is the obvious observation that the triangular matrices $H_- H_0^{-1}$ and H_+ have only 0's in their last column and row, respectivey. So assume that the result holds for $n \times n$ matrices, and that for an $(n+1) \times (n+1)$ matrix $P = H + H_- H_0^{-1} H_+$ we have the decomposition; note that in particular $P = P_n$ is invertible. Now, for the reciprocal implication, first observe that the decomposition of P gives a Borel decomposition of its principal submatrices, so in particular $P_{n-1} = G + G_- G_0^{-1} G_+$; using the observation above this immediately gives that $H_{n1} = P_{n1}$, $H_{1n} = P_{1n}$ and $H_{jj} = G_{jj} = |P_{jj}|/|P_{(j-1)(j-1)}|$; $0 \leqslant j < n$, and therefore

$$P_{nj} = H_{nj} H_{jj}^{-1}, \quad P_{jn} = H_{jn} H_{jj}^{-1},$$

and from this we get

$$P_{nn} = H_{nn} + \sum_{j=1}^{n} H_{nj} H_{jj}^{-1} H_{jn}.$$

Note that only the reciprocals of the diagonal elements appear. So the above equations are well defined and, therefore, all the new entries in the matrix H giving the decomposition of P are computed in terms of known quantities from P and of the decomposition of P_n. This proves the first and the last assertions for finite matrices.

To prove the statement about the diagonal elements of H_0, simply observe that from $P = (I + H_- H_0^{-1})(H_0 + H_+)$, using that both factors are triangular and the first one has 1's in the diagonal, we have that

$$|P| = \prod_{j=0}^{n} H_{jj} = \prod_{j=0}^{n} |P_j|/|P_{j-1}| = H_{nn}/|P_{n-1}|,$$

since this is a "telescopic" product.

Finally, it is now clear that, "passing to the limit as n tends to infinity"—and using again the uniqueness of the Borel decomposition—this argument gives a condition for the factorization of semi-infinite matrices H. □

For the case of bi-infinite matrices the situation is more complicated; nevertheless, we can state the following result.

Proposition 3. *Assume that P is a bi-infinite matrix such that $p_{ij} = 0$ whenever $i < 0$, $i < j$, and such that the semi-infinite submatrix D consisting of those p_{ij}, with $i, j \geqslant 0$, satisfies the above condition for Borel decomposition; then P itself admits a factorization.*

Also, if P admits a diagonal block representation

$$P=\begin{pmatrix} A & 0\\ 0 & D\end{pmatrix},$$

where A and D admit a Borel decomposition, then P admits a Borel decomposition.

Proof. For the first assertion, let

$$P=\begin{pmatrix} A & 0\\ C & D\end{pmatrix},$$

where by hypothesis we are assuming that D is a semi-infinite matrix admitting a Borel decomposition, and $A=A_-+A_0$. Then, if $H+H_-H_0^{-1}H_+$ gives the Borel decomposition of D, the Borel decomposition of P is given by the matrix $\tilde{H}$, where

$$\tilde{H}_-=\begin{pmatrix} A_- & 0\\ C & H_-\end{pmatrix},\quad \tilde{H}_0=\begin{pmatrix} A_0 & 0\\ 0 & H_0\end{pmatrix},\quad \tilde{H}_+=\begin{pmatrix} 0 & 0\\ 0 & H_+\end{pmatrix}.$$

The second assertion is clear. □

Remark. It is also immediate that the transpose of a matrix P having a Borel decomposition, $P=H+H_-H_0^{-1}H_+$, also has a Borel decomposition, since

$$P^{\mathrm{t}}=H^{\mathrm{t}}+H_+^{\mathrm{t}}H_0^{-1}H_-^{\mathrm{t}}=G+G_-G_0^{-1}G_+.$$

Example 1 (*Toda type lattices*). The tridiagonal symmetric matrices of the form $M=\Lambda^{\mathrm{T}}B+A+B\Lambda$ are related to the celebrated Toda lattice, describing a Hamiltonian system of (an infinite number of) particles in a line, each having interactions only with its immediate neighbors. Now, according to the previous results, we see that not every such matrix admits a Borel decomposition. However, if we start with a pair of diagonal matrices (Q_0, Q_1), Q_0 invertible, we can easily construct a Toda system admitting a Borel decomposition, by simply setting $H_-=\Lambda^{\mathrm{T}}Q_1$, $H_+=Q_1\Lambda$, $H_0=Q_0$, and $M=H+H_-H_0^{-1}H_+$.

More generally, we say that a Toda datum of order n is an $(n+1)$-tuple of diagonal matrices $(Q_0, Q_1, \ldots, Q_n)$, with Q_0 invertible. A Toda datum of order n allows us to construct a symmetric band matrix (a so-called *Jacobi matrix*), with $2n+1$ bands H, where $H_-=\sum_{k=1}^{n}\Lambda^{\mathrm{T}^k}Q_k$, $H_0=Q_0$, and $H_+=\sum_{k=1}^{n}Q_k\Lambda^k$, and therefore another Jacobi matrix $M=H+H_-H_0^{-1}H_+$. Observe that M has also $2n+1$ bands, since $H_-H_0^{-1}H_+$ involves only products of the form $\Lambda^{\mathrm{T}^i}\Lambda^j,\ 0\leqslant i,j\leqslant n$.

We can now describe the space of infinite matrices we are interested in, which we shall denote $\mathscr{M}^*$: First, let us denote by $\mathscr{G}$ the “relevant group” of matrices of $\mathscr{M}$ (ideally this should be the maximal subset of $\mathscr{M}$ closed under multiplication), then we will take our matrices to belong to $\mathscr{G}$. Moreover, we actually want to think of the coefficients as formal functions on an infinite set of time parameters, $\mathbf{t}=(t_1, t_2, \ldots)$; so, more precisely, we will consider $\mathscr{M}^*$ to be a trivial $\mathscr{G}$-bundle over the space of time parameters (which, leaving topological considerations aside and to fix ideas, we will take to be $\mathbb{R}((\mathbf{t}))$, the space of formal series in the time parameters), and the objects of interest will be sections of this bundle. For reasons that will be made clear later on, we add the final condition that $M_{\geqslant}(\mathbf{0}) = M_0(\mathbf{0}) + M_+(\mathbf{0}) = I$.

3. Frobenius integrability of matrix systems

The point in having the above factorization is that we can link the matrices in $\mathscr{M}$ to integrable systems, as we now show.

So, let us fix a lower triangular matrix (which will play the role of a Lax operator)

$$L = \Lambda + \sum_{k \geqslant 0} A_k \Lambda^{\mathrm{T}^k}. \tag{3}$$

As mentioned before, we think of the coefficients as formal functions (indeed, if necessary, formal power series) on an infinite set of time parameters, $\mathbf{t} = (t_1, t_2, \ldots)$, and consider the family of commuting "flows" $\partial/\partial t_k$. Then we define the *discrete KP hierarchy* as

$$\frac{\partial L}{\partial t_n} = [L^n_{\geqslant}, L], \quad n > 0, \tag{4}$$

where as before, for an infinite matrix T in $\mathscr{D}$, $T_{\geqslant}$ denotes its projection on the upper triangular part, i.e., on $\mathscr{D}_0 \oplus \mathscr{D}_+$.

We now attempt to describe this as a "Frobenius integrability" problem.

Our first step will be to introduce the (formal) connection 1-form

$$Z = \sum_{n \geqslant 1} L^n_{\geqslant} \mathrm{d}t_n. \tag{5}$$

The reason for the terminology is the following: as we mentioned, our space $\mathscr{M}^*$ of infinite matrices is a trivial $\mathscr{G}$-bundle over the space of time parameters (where $\mathscr{G}$ is the appropriate infinite dimensional group of matrices, as mentioned above); then, since we can formally think of $\mathscr{D}_+$ (or $\mathscr{D}_-$) as a Lie subalgebra of the Lie algebra of $\mathscr{G}$, we see that the $\mathscr{D}_+$-valued 1-form Z is in fact a connection in this bundle; furthermore, what we will now do is show that the discrete KP hierarchy (4) is equivalent to the flatness of this connection.

To see this, let us first show that we do have a "Zakharov–Shabat formalism" in this situation too.

Proposition 4. *If L satisfies the discrete KP equation* (4), *then the connection* 1-*form satisfies the Zakharov–Shabat equation*

$$\mathrm{d}Z = \tfrac{1}{2}[Z, Z]. \tag{6}$$

Proof. Observe that (4) implies that for any power of the operator L we have

$$\frac{\partial L^n}{\partial t_m} = \sum_{i=1}^{n} L^{i-1} \frac{\partial L}{\partial t_m} L^{n-k} = [L^m_{\geqslant}, L^n].$$

Therefore,

$$\frac{\partial L^n}{\partial t_m} - \frac{\partial L^m}{\partial t_n} = [L^m_{\geqslant}, L^n] + [L^n, L^m_{\geqslant}]. \tag{6$'$}$$

Now,

$$\begin{aligned}[L^m_{\geqslant}, L^n] + [L^n, L^m_{\geqslant}] &= [L^m, L^n] + [L^m_{\geqslant}, L^n_{\geqslant}] - [L^n_{-}, L^m_{-}] \\ &= [L^m_{\geqslant}, L^n_{\geqslant}] - [L^n_{-}, L^m_{-}].\end{aligned}$$

Thus, upon taking the upper triangular parts of both sides of (6$'$), we get

$$\frac{\partial L^n_{\geqslant}}{\partial t_m} - \frac{\partial L^m_{\geqslant}}{\partial t_n} = [L^m_{\geqslant}, L^n_{\geqslant}],$$

as desired. Now, recall that the Lie bracket of matrix valued 1-forms is defined using the commutator of the coefficients, and the exterior product of the forms; namely, if $A \otimes \alpha$, $B \otimes \beta$ are two decomposable matrix valued forms, then their Lie bracket is $[A, B] \otimes (\alpha \wedge \beta)$ (to avoid unnecessary cluttering of the notation we are omitting the tensor products in our formulas); therefore, using the connection Z, this system of equations can be nicely summarized into the concise single equation (6), which precisely says that Z is a flat connection on $\mathscr{M}^*$. These equivalent presentations are called the Zakharov–Shabat or “zero-curvature” equations of the system.

Thus, solutions of the discrete KP hierarchy imply solutions of the Zakharov–Shabat system; to get the converse implication, we shall show next the existence of a *Sato–Wilson dressing operator*; that is an invertible matrix $S \in \mathscr{G}$ such that $S = I + S_-$, $L = S\Lambda S^{-1}$, and such that (4) is equivalent to the system

$$\frac{\partial S}{\partial t_n} = L^n_{-} S. \tag{7}$$

Now, existence of dressing operators follows in the usual way, from recursively solving a system of equations for the dressing operator. The point is that not every dressing operator will solve (7). Nevertheless the ambiguity in determining dressing operators clearly lies in the group of operators with constant coefficients (i.e., those matrices commuting with Λ), and this allows us to find Sato–Wilson operators. □

Proposition 5. *There exist Sato–Wilson operators.*

Proof. Let T be any dressing operator for L, where L is a solution of (4), fix some $n \in \mathbb{N}$, and consider the gauge transform of L^n_{-},

$$P = -T^{-1} L^n_{-} T - T^{-1} \frac{\partial T}{\partial t_n}.$$

Then P commutes with Λ, because

$$[P,\Lambda] = T^{-1}[TPT^{-1}, T\Lambda T^{-1}]T$$
$$= T^{-1}[-L_-^n - \frac{\partial T}{\partial t_n}T^{-1}, L]T$$
$$= T^{-1}\left(\frac{\partial L}{\partial t_n} - \left[\frac{\partial T}{\partial t_n}T^{-1}, L\right]\right)T = 0.$$

Now, $P = \sum_{k>0} p_k \Lambda^{\mathrm{T}^k}$ can always be written as $\partial C/\partial t_n C^{-1}$ for some $C = I + \sum_{k>0} c_k \Lambda^{\mathrm{T}^k}$, since this is equivalent to solving the system of ODEs

$$p_k + \sum_{i+j=k;\ i,j>0} p_i c_j^{(-i)} = \frac{\partial c_k}{\partial t_n},$$

which is clearly recursively solvable (observe that we can also think of $P\mathrm{d}t_n$ as a flat connection on a 1-dimensional bundle). Moreover, since $[P,\Lambda] = 0 \iff p_k = p_k^{(-1)}$, which means that each diagonal p_k is a scalar multiple of the identity I, it follows that we can choose C so that $[C,\Lambda] = 0$; for instance, we have $p_1 = \partial c_1/\partial t_n$, so we can choose $c_1 = c_1^{(-1)}$, and then proceed recursively.

Thus $S = TC$ is a dressing operator, and is in fact Sato–Wilson, because, by the definition of P, we have

$$\frac{\partial S}{\partial t_n} = \frac{\partial T}{\partial t_n}C + T\frac{\partial C}{\partial t_n} = -TPC - L_-^n TC + TPC = -L_-^n S$$

as desired. □

Again, the system of equations (7) can be recast into the single equation

$$\mathrm{d}S = Z_c S, \tag{8}$$

where $Z_c = -\sum_{n\geqslant 1} L_-^n \mathrm{d}t_n$ is the so-called *conjugate* (or *complementary*) connection.

Next, consider the "trivial" connection: $\Omega = \sum_{n\geqslant 1} \Lambda^n \mathrm{d}t_n$; then we have:

Proposition 6. *Let S be a Sato–Wilson operator and Ω as before. Then* $\mathrm{d}\Omega = \frac{1}{2}[\Omega,\Omega]$, *and*

$$S\Omega S^{-1} + \mathrm{d}S\, S^{-1} = Z.$$

Proof. The first assertion is clear, since Ω has only constant coefficients, so both sides are zero. For the second simply use the definition of Ω, and the fact that $S\Lambda^n S^{-1} = L^n$. □

Observe that the proposition says that Ω is a flat connection, and Z is a gauge transform of this flat connection. From the theory of connections we know that this will imply that Z is also flat (which is just the Zakharov–Shabat equation), and hence that there will exist a matrix Y such that $Z = \mathrm{d}Y\,Y^{-1}$. However, the proof is quite simple, so we give it.

Proposition 7. *If Z satisfies the Zakharov–Shabat equation, then there exists a matrix Y such that*

$$Z = \mathrm{d}Y\,Y^{-1}. \tag{8$'$}$$

Proof. First we prove that this holds for Ω; namely $\Omega = \mathrm{d}U\,U^{-1}$. But, if we let $U = I + \sum_{i>0} u_i \Lambda^i$, this is equivalent to solving for each n the following system of equations:

$$\Lambda^n + \sum_{k>0} u^{(-n)} \Lambda^{n+i} = \sum_{0<i} \frac{\partial u_i}{\partial t_n} \Lambda^i,$$

which is clearly solvable (for instance, $u_n = \sum_{1\leqslant i\leqslant n} t_i I$ is a solution).

Now take a Sato–Wilson operator S and let $Y = SU$; then $Y^{-1} = U^{-1}S^{-1}$, and $\mathrm{d}Y = \mathrm{d}S\,U + U\,\mathrm{d}S$, so that

$$\mathrm{d}Y\,Y^{-1} = \mathrm{d}S\,S^{-1} + S\mathrm{d}U\,U^{-1}S^{-1} = \mathrm{d}S\,S^{-1} + S\Omega S^{-1} = Z$$

by Proposition 6. □

Indeed, the first part of the proposition may be proved in a different way, by noting that, since Ω is constant, we have the formal solution $U(\mathbf{t}) = \exp(\sum_{n=1}^{\infty} t_n \Lambda^n)$. The reader can check that both computations give the same answer.

The main result on the integrability of the discrete KP system may now be stated as follows:

Proposition 8. *Each solution of the discrete KP system* (4) *yields a solution of the equation*

$$\mathrm{d}U = \Omega U, \tag{9}$$

where U is a section of M^*, and conversely. In this sense, the two systems are equivalent.

Proof. If $U = S^{-1}Y$ is the Borel decomposition of a solution U of (9), define $L = S\Lambda S^{-1}$, $Z = \mathrm{d}Y\,Y^{-1}$, and $Z_c = \mathrm{d}S\,S^{-1}$, then

$$\begin{aligned} S\Omega S^{-1} &= S\,\mathrm{d}U\,U^{-1}S^{-1} \\ &= S\,\mathrm{d}(S^{-1}Y)Y^{-1}SS^{-1} \\ &= S\,\mathrm{d}S^{-1} + \mathrm{d}Y\,Y^{-1} \\ &= \mathrm{d}Y\,Y^{-1} - \mathrm{d}S\,S^{-1} \\ &= Z - Z_c. \end{aligned}$$

Since by construction $S\Omega S^{-1} = \sum_n L^n \mathrm{d}t_n$, and the above formula gives its decomposition into upper triangular and strictly lower triangular parts, we conclude that $Z_c = -\sum_n L_-^n \, \mathrm{d}t_n$. So S is a Sato–Wilson operator, and L is then a solution of the discrete KP system.

For the reciprocal, given that L is a solution of (4), with associated flat connection $Z = \mathrm{d}Y\, Y^{-1}$, let S be a Sato–Wilson operator and define $U = S^{-1}Y$. Then

$$\begin{aligned} \mathrm{d}U &= \mathrm{d}S^{-1}\, Y + S^{-1}\, \mathrm{d}Y \\ &= -S^{-1}\, \mathrm{d}S\, S^{-1}Y + S^{-1}\, \mathrm{d}Y \\ &= S^{-1}(Z - \mathrm{d}S\, S^{-1})Y \\ &= S^{-1}(S\Omega S^{-1})Y \\ &= \Omega U, \end{aligned}$$

where the next to last equality comes from Proposition 6. □

Remark. From what has been said, we have the existence and uniqueness result of solutions to the initial value problem for (9): this is in a sense just the content of Proposition 7, since every solution of $\mathrm{d}U = \Omega U$ is of the form

$$\hat{U}(\mathbf{t}) = \exp\left(\sum_{n=1}^{\infty} t_n \Lambda^n\right) U(0) = \mathrm{e}^{\Sigma(\mathbf{t})} U_0,$$

where $U_0 = U(0)$ is the initial condition. Furthermore, observe that this actually holds for solutions that are not necessarily of the form $I + U_+$ since, in the general case, if we let

$$U = \sum_{k>0} u_k \Lambda^k + u_0 + \sum_{k>0} u_{-k} \Lambda^{\mathrm{T}^k},$$

Eq. (9) is equivalent to saying that the system of equations

$$\frac{\partial u_n}{\partial t_m} = \Lambda^n u_{n-m} \Lambda^{\mathrm{T}^n}$$

has a solution for all m, n. This is a problem of solving a system of ordinary differential equations, and the system is consistent, since this is the same as saying that Ω is flat.

Now, the point is that this solves in fact the initial value for the discrete KP system (4) as follows: Clearly, if $U(\mathbf{t})$ is a decomposable solution to (9), then writing $U = S^{-1}Y$ as in the proposition we see that, since $Y(\mathbf{0}) = I$, so that $U(\mathbf{0}) = S^{-1}(\mathbf{0})$, $L = S\Lambda S^{-1}$ satisfies (4) and its value at $\mathbf{0}$ is $L_0 = S_0 \Lambda S_0^{-1}$, where $S_0 = S(\mathbf{0})$. But conversely, if $L(\mathbf{0}) = L_0 = S_0 \Lambda S_0^{-1}$, then we can assume that $S_0 = S(\mathbf{0})$, where S is Sato–Wilson, because changing the dressing operator will not change the initial value L_0. Therefore, writing $Z = \mathrm{d}Y\, Y^{-1}$, $Z_c = \mathrm{d}S\, S^{-1}$, then the proposition tells us that $U = S^{-1}Y$ is a solution of (9), and since $\mathrm{e}^{\Sigma(\mathbf{0})} = I = Y(\mathbf{0})$, we get that $U(\mathbf{0}) =$

S_0^{-1} as desired. (And it is exactly at this point that the condition $M_{\geqslant}(\mathbf{0}) = I$ for the sections of $\mathscr{M}^*$ is needed.)

Example 2. Let us use our results to explicitly exhibit a matrix integrable system.

As initial data we take the very simple matrix

$$S_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ a & 1 & 0 & 0 & \cdots \\ c & b & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and the solution to (9) is then

$$U(\mathbf{t}) = \exp\left(\sum_{n \geqslant 0} t_n \Lambda^n\right) S_0^{-1}.$$

A direct computation shows that

$$U(\mathbf{t}) = \begin{pmatrix} 1 + at_1 + c\left(\frac{t_1^2}{2} + t_2\right) & t_1 + b\left(\frac{t_1^2}{2} + t_2\right) & \left(\frac{t_1^2}{2} + t_2\right) & \cdots & \cdots \\ a + ct_1 & 1 + bt_1 & t_1 & \left(\frac{t_1^2}{2} + t_2\right) & \cdots \\ c & b & 1 & t_1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We now try to write $U(\mathbf{t}) = S^{-1}Y$, where S^{-1} is a Borel decomposable lower triangular matrix. It is not hard to see that, with the special choice of initial conditions made, everything is determined by the 3×3 upper left corner matrix, which to avoid unnecessary extra notations we still denote by U:

$$U = \begin{pmatrix} 1 + at_1 + c\left(\frac{t_1^2}{2} + t_2\right) & t_1 + b\left(\frac{t_1^2}{2} + t_2\right) & \left(\frac{t_1^2}{2} + t_2\right) \\ a + ct_1 & 1 + bt_1 & t_1 \\ c & b & 1 \end{pmatrix};$$

and so, in what follows, we will just work with 3×3 matrices, corresponding to the upper left corners of the semi-infinite matrices of the original system. (Indeed, the remaining non-zero terms of S^{-1} will be just 1's along the main diagonal.) We also observe that U is in fact a Wronskian matrix with respect to the variable t_1, and we relabel it as

$$\begin{pmatrix} \alpha & \beta & \left(\frac{t_1^2}{2} + t_2\right) \\ \gamma & \delta & t_1 \\ c & d & 1 \end{pmatrix},$$

to simplify the writing.

Thus, according to Proposition 3, we can now write $S^{-1} = I + H_- H_0^{-1}$, where, in fact, after a short computation we can write the explicit expressions

$$H_- = \begin{pmatrix} 0 & 0 & 0 \\ \gamma & 0 & 0 \\ c & \frac{b\alpha - c\beta}{\alpha} & 0 \end{pmatrix}, \quad H_0 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{\varDelta}{\alpha} & 0 \\ 0 & 0 & \frac{|U|}{\varDelta} \end{pmatrix};$$

as before $|\cdot|$ denotes the determinant, and we have written $\varDelta = \alpha\delta - \beta\gamma$. Observe that the diagonal elements of H_0^{-1} are rational functions in the variables t_1, t_2, so we have singularities of the system at the zeros of the denominators. Moreover, writing

$$S^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ A & 1 & 0 \\ C & B & 1 \end{pmatrix},$$

where we have set

$$A = \frac{\gamma}{\alpha}, \quad B = \frac{b\alpha - c\beta}{\alpha\delta - \beta\gamma}, \quad C = \frac{c}{\alpha},$$

a direct computation then gives

$$S = \begin{pmatrix} 1 & 0 & 0 \\ -A & 1 & 0 \\ AB - C & -B & 1 \end{pmatrix},$$

and so we get as a solution

$$L = S\varLambda S^{-1} = \begin{pmatrix} A & 1 & 0 \\ -A^2 + C & -A + B & 1 \\ (A^2 - C)B - AC & (AB - C) - B^2 & -B \end{pmatrix}. \tag{10}$$

Reverting to semi-infinite matrices, the solution of (4) we get is

$$L = \begin{pmatrix} A & 1 & 0 & 0 & \cdots \\ -A^2 + C & -A + B & 1 & 0 & \cdots \\ (A^2 - C)B - AC & (AB - C) - B^2 & -B & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{11}$$

Remark. Although to simplify the explicit computations we have chosen a system that reduces to a finite dimensional problem, we would like to stress the fact that the method applies to truly infinite dimensional systems.

On the other hand, it is also an instructive exercise to verify directly that the matrix obtained in (10) indeed solves (4). This is more easily done by using the fact that, by construction, S is a Sato–Wilson operator, so it satisfies (7), namely

$$\frac{\partial S}{\partial t_n} = -L_-^n S.$$

In our example this leads to the equations

$$\frac{\partial S}{\partial t_1} = -L_- S, \quad \frac{\partial S}{\partial t_2} = -L_-^2 S,$$

which are equivalent to two systems of three equations. For instance, the former is equivalent to

$$\frac{\partial A}{\partial t_1} = -A^2 + C, \quad \frac{\partial (AB - C)}{\partial t_1} = B(AB - C), \quad \frac{\partial B}{\partial t_1} = (AB - C) - B^2;$$

then, again for instance, to verify the first or the last of these equations, one observes that A and B are logarithmic derivatives,

$$A = \frac{\frac{\partial \alpha}{\partial t_1}}{\alpha}, \quad B = \frac{\frac{\partial \Delta}{\partial t_1}}{\Delta},$$

which makes the assertions easy to deal with; the remaining equations are similarly handled, and the reader can check that in fact $L^3 = 0$, which is not entirely unexpected since, in this example, there is no non-trivial dependence upon t_n for $n > 2$.

Let us finally show that L, given by (3), satisfies a string type equation. For this, we fix a Sato–Wilson operator S of L, and consider the family of flows

$$\tilde{A}_n = \frac{\partial}{\partial t_n} - \Lambda^n = \partial_n - \Lambda^n$$

together with their dressed counterparts,

$$A_n = S\tilde{A}_n S^{-1},$$

as well as the operator

$$\tilde{M} = \epsilon + \sum_{k \geqslant 1} k t_k \Lambda^{k-1},$$

(where $\epsilon = \operatorname{diag}(1, 2, 3, \ldots)\Lambda^{\mathrm{T}}$ if we are in $\mathscr{D}^{\infty}$, or $\epsilon = \operatorname{diag}(\ldots, -1, 0, 1, 2, \ldots)\Lambda^{\mathrm{T}}$ if we are in $\mathscr{D}^{\infty}_{-\infty}$), and also its gauge transform $M = S\tilde{M}S^{-1}$. Observe that $[\Lambda, \epsilon] = I$ is immediate from the definition of ϵ.

We have:

Proposition 9. *Denote* $B_n = L^n_{\geqslant}$. *Then* $A_n = \partial_n - B_n$. *As a consequence*, $[A_n, L] = 0 = [A_n, M]$, *and* $[L, M] = I$.

(It is the last equation that is sometimes called a *string equation*.)

Proof. For the first assertion, observe that

$$S \circ \partial_n \circ S_{-1} = S(\partial_n S^{-1}) + \partial_n$$

(for the sake of clarity, we have made here explicit that on the left-hand side we have the composition of operators, whereas on the right-hand side the operator ∂_n *acts* on the coefficients of S^{-1}). Therefore, since S is Sato–Wilson,

$$\begin{aligned} A_n &= S(\partial_n S^{-1}) + \partial_n - L^n \\ &= -(\partial S)S^{-1} + \partial_n - L^n \\ &= L^n_- + \partial_n - L^n \\ &= \partial_n - B_n. \end{aligned}$$

It follows immediately that

$$[A_n, L] = [S\tilde{A}_n S^{-1}, S\Lambda S^{-1}] = S[\tilde{A}_n, \Lambda]S^{-1} = 0,$$

thus proving the first of the last equalities stated.

Since $[\Lambda, \epsilon] = I$, by induction we then have $\Lambda^n \epsilon = n\Lambda^{n-1} + \epsilon\Lambda^n$, so that $[\Lambda^n, \epsilon] = n\Lambda^{n-1}$. Obviously we also have

$$[\partial_n, \epsilon] = 0 = \left[\Lambda^n, \sum_{k\geqslant 1} kt_k \Lambda^{k-1}\right],$$

therefore,

$$\begin{aligned} [\tilde{A}_n, \tilde{M}] &= -[\Lambda^n, \epsilon] + \left[\partial_n, \sum_{k\geqslant 1} kt_k \Lambda^{k-1}\right] \\ &= -n\Lambda^{n-1} + \sum_{k\geqslant 1} kt_k \Lambda^{k-1}\partial_n + n\Lambda^{n-1} - \sum_{k\geqslant 1} kt_k \Lambda^{k-1}\partial_n = 0. \end{aligned}$$

To end the proof it now suffices to notice that

$$[L, M] = S[\Lambda, \tilde{M}]S^{-1} = S[\Lambda, \epsilon]S^{-1} = I. \qquad \square$$

Acknowledgement

The first author wishes to warmly thank the Mexican scientific authorities, and particularly CIMAT, where this work was written, for its hospitality and support. Actually, both of them are glad to acknowledge the continuing support to their joint research of their respective co-author's home institution.

References

[1] M. Adler, Group factorization, moment matrices and Toda lattices, Int. Math. Res. Notices 12 (1997) 556–572.

[2] M. Adler, P. Van Moerbeke, Vertex operator solutions to the discrete KP hierarchy, Commun. Math. Phys. 203 (1999) 185–210.

[3] M. Adler, P. Van Moerbeke, Generalized orthogonal polynomials, discrete KP and Riemann–Hilbert problems, Commun. Math. Phys. 207 (1999) 589–620.
[4] M. Guest, Harmonic Maps, Loop Groups, and Integrabe Systems, London Mathematical Society. Student Texts, vol. 38, 1997.
[5] M. Mulase, Complete integrability of the Kadomtsev–Petviashvili equation, Adv. Math. 54 (1984) 57–66.